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THE INTRODUCTION OF ALGEBRAIC OPERATIONS ON THE SET OF TRAJECTORIES OF A NON-LINEAR CONTROL SYSTEM*

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The problem of the stabilisation of a mechanical control system is considered. The controls ensuring the stability in the large are determined using new algebraic operations. An example of the control of a mechanical system by means of two moments is given.

1. Introduction. Consider the following control system (an asterisk denotes transpostion):

$$x^{\cdot} = f(x, u);\ x = (x_1, \ldots, x_n)^*,\ u = (u_1, \ldots, u_m)^* \tag{1.1}$$

We shall call the solution $x(t)$ of (1.1) corresponding to the control $u(t)$, the trajectory of (1.1) and denote by W the set of all pairs $x(t), u(t)$ of vector functions sayisfying (1.1). We shall assume that the limits of variation of t, usually infinite, are determined by the problem in question (they need not be specified). The aim of this paper is to introduce abstract laws of addition $\oplus$ and multiplication by a number $\odot$, applied to the elements of the set W associated with the dynamics of the system.

The following linear system serves as a special case of system (1.1):

$$x^{\cdot} = Ax + Bu \tag{1.2}$$

where A and B are the $n \times n$ and $n \times m$ matrices respectively. The operation of adding two trajectories $x'(t)$ and $x''(t)$ is determined by those operations in Euclidean space

$$x'''(t) = x'(t) + x''(t),\quad \lambda x(t) \tag{1.3}$$

therefore we can write for (1.2) $\oplus \equiv +$, $\odot \equiv \cdot$. The laws (1.3) for the trajectories have the corresponding laws of composition for the controls

$$u'''(t) = u'(t) + u''(t),\quad \lambda u(t) \tag{1.4}$$

The trajectory $x'''(t)$ with initial condition $x'''(0) = x'(0) + x''(0)$, corresponds to the trajectories $x'(t)$ and $x''(t)$ with initial conditions $x'(0)$ and $x''(0)$, i.e. the initial conditions obey the same laws of composition as the trajectories.

The laws of composition (1.3), (1.4) for the basis of the analysis and synthesis of the linear automatic control systems. For this reason it is natural that we should try to extend them in some manner to non-linear systems. Two different approaches are possible. In the first method we derive the laws of composition on the set W from R^n, R^m by induction, i.e. we determine them according to (1.3). This leads to linearization methods. The second approach involves changing to new laws of composition $\oplus, \odot$, and connecting them with the laws of dynamics (1.1). We shall adopt the second approach.

On introducing the laws $\oplus, \odot$ we find that system (1.1) is linear, though not in the sense of the operations $+, \cdot$ (1.3) induced from R^n, but in the sense of the new laws $\oplus, \odot$, and not in an approximate manner as in linearization methods, but exactly. The functional representation entry $(u(t))$ - exit $(x(t))$ becomes linear (not approximately, but exactly) for the laws of composition $\oplus, \odot$. The principle of superposition holds: the trajectory $x'(\cdot) \oplus x''(\cdot)$ corresponds to the control $u'(\cdot) \oplus u''(\cdot)$, and similarly, $\lambda \odot x(\cdot)$ corresponds to $\lambda \odot u(\cdot)$.

A considerable numer of corollaries can be drawn from this, connected e.g. with transferring the methods of analysis and synthesis of the linear systems to the non-linear systems, but in terms of the operations $\oplus, \odot$. The method of synthesising the regulator of a non-linear system with prescribed requirements concerning the dynamics given below, represents one such application and is based on non-linear continuation of the regulator of the linearized system, from the matching neighbourhood to the whole phase space of the system. The proposed approach can be used with a systems of the form (3.1), (3.11) and condition (3.12), when system (5.8) has a unique solution in $p_1, \ldots, p_m$. The exact conditions are given in Theorem 7.1.

To avoid possible misunderstanding, we will assume without discussion that from now on

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$$1 \leqslant m < n \tag{1.7}$$

everywhere.

2. New law $\oplus$ of addition in R^n. We shall introduce the law of composition $\oplus_x$ on the set R^n in the form of a differentiable vector function $\varphi(x', x'')$ which places two vectors $x', x'' \in R^n$ in 1:1 correspondence with the third vector $x''' \in R^n$. We write this as follows:

$$x''' = \varphi(x', x'') = x' \oplus_x x'' \tag{2.1}$$

We seek the law of composition $\oplus_u$ for the control $u(t)$ in the form of the function

$$u''' = \psi(x', u', x'', u'') = u' \oplus_u u'' \tag{2.2}$$

which places the control vector $u''' \in R^m$ in 1:1 correspondence with every two states x', x'' and control vectors u', u''.

Introduction of the new laws $\oplus_x, \oplus_u$ simplifies the study of the system, provided that the laws $\oplus_x, \oplus_u$ induce, for the non-linear system, the laws of addition of trajectories and controls in exactly the same manner as the law of addition $+$ in R^n induces the addition of trajectories (1.3) and controls (1.4) in the linear system. Therefore, we shall require that the laws $\oplus_x, \oplus_u$ do not cause departure from the set W. This means that the equations

$$x^{\cdot\prime}(t) = f(x'(t), u'(t)), \quad x^{\cdot\prime\prime}(t) = f(x''(t), u''(t)) \tag{2.3}$$

must imply

$$x^{\cdot\prime\prime\prime}(t) = f(x''', u''') \tag{2.4}$$

On the other hand, according to (2.1), (2.3) we have

$$x^{\cdot\prime\prime\prime}(t) = \varphi_{x'} f(x', u') + \varphi_{x''} f(x'', u'') \tag{2.5}$$

where $\varphi_{x'}, \varphi_{x''}$ are the corresponding matrices of the partial derivatives. From (2.4) and (2.5) we obtain the fundamental equation in partial derivatives

$$f(\varphi(x', x''), u''') = \varphi_{x'}(x', x'') f(x', u') + \varphi_{x''}(x', x'') f(x'', u'') \tag{2.6}$$

The equation reflects the fact that the new laws of combination $\oplus_x, \oplus_u$ introduced as the mappings $R^n \times R^n \to R^n$, $R^n \times R^m \times R^n \times R^m \to R^m$, induce the laws of composition for the trajectories. This means that if $x'(t)$, $x''(t)$ satisfy on $a < t < b$ equation (1.1) for $u'(t)$ and $u''(t)$ respectively, then the vector function $x'''(t)$ defined for every t by the equation

$$x'''(t) = \varphi(x'(t), x''(t)) \tag{2.7}$$

satisfies (1.1) after substituting into it of the vector function $u'''(t)$ given for every t by the formula

$$u'''(t) = \psi(x'(t), u'(t), x''(t), u''(t)) \tag{2.8}$$

When it is clear from the text which vectors do combine, whether x or u, we can write instead of $\oplus_x, \oplus_u$, simply $\oplus$.

3. Solution of the equation defining the laws $\oplus_x, \oplus_u$. We shall limit ourselves to considering a system, linear with respect to the control

$$x^\cdot = X^\circ(x) + Y^\circ(x)u; \quad X^\circ(x) = \left\| \begin{matrix} F(x) \\ Z(x) \end{matrix} \right\|, \quad Z(x) = \left\| \begin{matrix} Z_1(x) \\ \cdots\cdots \\ Z_{n-m}(x) \end{matrix} \right\| = \left\| \begin{matrix} X^\circ_{m+1}(x) \\ \cdots\cdots \\ X_n^\circ(x) \end{matrix} \right\|, \quad F(x) = \left\| \begin{matrix} X_1^\circ(x) \\ \cdots\cdots \\ X_m^\circ(x) \end{matrix} \right\| \tag{3.1}$$

where $X^\circ(x)$ is a column and $Y^\circ(x)$ is an $n \times m$-matrix. The domain of variation of u is the whole space R^m. Let us denote by Γ_x the image of the mapping $u \to Y^\circ(x)u$, i.e. $\Gamma_x = Y^\circ(x) R^m$.

Theorem 3.1. Let $\varphi(x', x'')$ be a continuously differentiable function defining on R^n the structure of the commutative group by the formula (2.1). Then we have everywhere

$$\operatorname{rank} \varphi_{x'}(x', x'') = \operatorname{rank} \varphi_{x''}(x', x'') = n \tag{3.2}$$

Proof. Let x_0'' be an element inverse to x'' with respect to addition $\oplus$. Then $x \oplus x'' \oplus x_0'' = x$, which we can write in the form $\varphi(\varphi(x, x''), x_0'') = x$. Differentiating this expression with respect to x, we obtain

$$\varphi_{x'}\varphi_x(x, x'') = E$$

where E is the unit matrix. Expression (3.2) follows from this.

We can write equation (2.6) for system (3.1) in the form

$$X^\circ(\varphi) + Y^\circ(\varphi)\psi = \varphi_{x'}[X^\circ(x') + Y^\circ(x')u'] + \varphi_{x''}[X^\circ(x'') + Y^\circ(x'')u''] \tag{3.3}$$

Theorem 3.2. To find a continuously differentiable function $\varphi(x', x'')$ and a function $\psi(x', u', x'', u'')$, which will satisfy equation (3.3) for all x', x'', u', u'', it is necessary that for all $x', x'' \in R^n$.

$$\varphi_{x'}\Gamma_{x'} \subset \Gamma_\varphi \tag{3.4}$$

$$\varphi_{x'}X^\circ(x') + \varphi_{x''}X^\circ(x'') - X^\circ(\varphi) \in \Gamma_\varphi \tag{3.5}$$

If φ defines on R^n the structure of the commutative group and we have, for all $x \in R^n$

$$\operatorname{rank} Y^\circ(x) = m \tag{3.6}$$

then (3.4) is equivalent to the equation

$$\varphi_{x'}\Gamma_{x'} = \Gamma_\varphi \tag{3.7}$$

Conversely, if $\varphi(x', x'')$ is a continuously differentiable commutative function for which conditions (3.4), (3.5) hold for all $x', x'' \in R^n$, then a function $\psi(x', u', x'', u'')$, can be found such that ψ, φ satisfy equation (3.3) for all $x', x'' \in R^n$, $u', u'' \in R^m$.

Proof. Let $\varphi(x', x'')$ be a differentiable function satisfying (3.3) together with some function ψ. Assuming in (3.3) $u' = u'' = 0$ we obtain, for some $u_0 = \psi(x', 0, x'', 0) \in R^m$

$$X^\circ(\varphi) + Y^\circ(\varphi)u_0 = \varphi_{x'}X^\circ(x') + \varphi_{x''}X^\circ(x'') \tag{3.8}$$

which again means (3.5). Let us subtract (3.8) from (3.3)

$$Y^\circ(\varphi)(\psi - u_0) = \varphi_{x'}Y^\circ(x')u' + \varphi_{x''}Y^\circ(x'')u'' \tag{3.9}$$

For equation (3.9) to have a solution in ψ for any u', u'' it is necessary and sufficient that

$$\Gamma_\varphi \supset \varphi_{x'}\Gamma_{x'} + \varphi_{x''}\Gamma_{x''} \tag{3.10}$$

and this, in turn, is equivalent to two inclusions

$$\varphi_{x'}\Gamma_{x'} \subset \Gamma_\varphi, \quad \varphi_{x''}\Gamma_{x''} \subset \Gamma_\varphi$$

If (3.6) holds, then $\dim \Gamma_x = m$ for any x. If moreover φ defines a group, then according to Theorem 3.1 we have (3.2). Then $\dim \varphi_{x'}\Gamma_{x'} = m$ and the inclusion (3.4) becomes (3.7). Conversely, (3.5) implies that u_0 exists satisfying (3.8). For the commutative function $\varphi(x', x'')$ (3.4) implies two last inclusions, and from this it follows that (3.9) has a solution in ψ. Combining (3.8) and (3.9), we obtain (3.3).

Let us now consider a system with constant coefficients acted upon by the controls

$$Y^\circ = \begin{Vmatrix} Y \\ 0 \end{Vmatrix} = \text{const}; \quad Y = \begin{Vmatrix} Y_{11}, \ldots, Y_{1m} \\ Y_{m1}, \ldots, Y_{mm} \end{Vmatrix} \tag{3.11}$$

Theorem 3.3. Let

$$\operatorname{rank} Y = m \tag{3.12}$$

Then, in the case of a continuously differentiable function $\varphi(x', x'')$ condition (3.7) is equivalent to the statement that when $i > m$, the function φ_i is independent of $x_1', \ldots, x_m'$, $x_1'', \ldots, x_m''$

$$\varphi_i = \varphi_i(x_{m+1}', \ldots, x_n', x_{m+1}'', \ldots, x_n''), \quad m+1 \leqslant i \leqslant n \tag{3.13}$$

When (3.13) holds, the inclusion (3.5) is equivalent to

$$\varphi_{x'}^{(2)}Z(x') + \varphi_{x''}^{(2)}Z(x'') = Z(\varphi) \tag{3.14}$$

$$\varphi_{x'}^{(2)} = \begin{Vmatrix} \dfrac{\partial\varphi_{m+1}}{\partial x_{m+1}'} & \cdots & \dfrac{\partial\varphi_{m+1}}{\partial x_n'} \\ \cdot & \cdots & \cdot \\ \dfrac{\partial\varphi_n}{\partial x_{m+1}'} & \cdots & \dfrac{\partial\varphi_n}{\partial x_n'} \end{Vmatrix}$$

Proof. By virtue of (3.11) and (3.12) $\Gamma_x = \text{const}$ is a linear subspace stretched over the first m coordinate axes. From this it follows that for (3.7) to hold it is necessary and sufficient for the $m+1$-th to n-th coordinates of the vectors $\varphi_{x'}Y^\circ u$ to be zero for all $u \in R^m$. This is equivalent to requiring that all elements of the matrix $\varphi_{x'}Y^\circ$ lying in the $m+1$-th to n-th rows be zero

$$\sum_{j=1}^{m} \frac{\partial\varphi_i}{\partial x_j} Y_{jl}^\circ = 0, \quad i = m+1, m+2, \ldots, n, \quad l = 1, 2, \ldots, m$$

By virtue of (3.12), this is equivalent to requiring that $\partial\varphi_i/\partial x_j \equiv 0$ for all x' and x'', for $i = m+1, m+2, \ldots, n$, $j = 1, 2, \ldots, m$, and this in turn is equivalent to (3.13), which implies the equivalence of (3.5) and (3.14).

We shall define the law of addition $\oplus_x$ over the coordinates with indices $m+1$ to n, in the usual manner

$$\varphi_i(x', x'') = x_i' + x_i'', \quad m+1 \leqslant i \leqslant n \tag{3.15}$$

In this case equation (3.14) takes the form

$$Z(x') + Z(x'') = Z(\varphi), \quad Z(x) \in R^{n-m} \tag{3.16}$$

The latter expression, together with (3.15), defines $2(n-m)$ equations for determining φ. It is preferable that they should give φ uniquely, although this is usually not the case when $n > 2(n-m)$. Therefore, when $n > 2(n-m)$, we complement system (3.16) with $2m-n$ equations of the form e.g. (3.15), in order to make φ uniquely definable. This corresponds to considering the following vector equation:

$$X(x') + X(x'') = X(\varphi), \quad X(x) \in R^n \tag{3.17}$$

in which the first $n-m$ equations are identical with (3.16) and the last m equations are identical with (3.15)

$$X_i(x) = Z_i(x) = X_{m+i}^\circ(x), \quad 1 \leqslant i \leqslant n-m \tag{3.18}$$

$$X_i(x) = x_i, \quad m+1 \leqslant i \leqslant n \tag{3.19}$$

If $n > 2(n-m)$, we specify the components $X_i(x)$, $n-m+1 \leqslant i \leqslant m$ in a suitable manner, e.g. $X_i(x) = x_i$. The situation $n \leqslant 2(n-m)$ is also permitted. In this case we determine $X_i(x)$, $m+1 \leqslant i \leqslant n$, according to (3.19) and the components $X_i(x)$, $1 \leqslant i \leqslant m$ according to (3.18).

Theorem 3.4. Let

1) equation (3.17) have a solution $\varphi(x', x'')$ for any $x', x'' \in R^n$
2) the equation $X(x) = X(x')$ implies that $x = x'$
3) the following relation holds:

$$X(0) = 0 \tag{3.20}$$

4) equation

$$X(x) + X(x^\circ) = 0 \tag{3.21}$$

has a solution $x^\circ(x)$ for all x.

Then $\varphi(x', x'')$ imposes on R^n the structure of a commutative group.

Proof. According to (2.1) and (3.17) we have

$$X(x' \oplus x'') = X(x') + X(x'') = X(x'') + X(x') = X(x'' \oplus x')$$

This, together with condition 2 of the theorem yields $x' \oplus x'' = x'' \oplus x'$. We obtain in the same manner

$$X((x' \oplus x'') \oplus x''') = X(x' \oplus x'') + X(x''') = X(x') + X(x'') + X(x''') = X(x') + X(x'' \oplus x''') = X(x' \oplus (x'' \oplus \oplus x'''))$$

This, together with condition 2 of the theorem shows that the operation $\oplus$ is associative $(x' \oplus x'') \oplus x''' = x' \oplus (x'' \oplus x''')$. Since by virtue of (3.20) $X(x \oplus 0) = X(x) + X(0) = X(x)$, from which according to the condition we have $2x \oplus 0 = x$, it follows that 0 represents a null element also for the operation $\oplus$. From (3.21) it follows that $x^\circ(x)$ is the inverse of x. Indeed, $X(x) + X(x^\circ) = X(x \oplus x^\circ) = 0 = X(0)$. Then from condition 2 of the theorem we obtain $x \oplus x^\circ = 0$.

When $n \geqslant 2(n-m)$, the solution of (3.17) satisfies (3.16). When $n < 2(n-m)$ we use this fact as an assumption. In this case the theorem reduces the construction of the operation $\oplus_x$ to solving equation (3.17).

We obtain the law of composition (2.2) for the control vectors of system (3.1), (3.11), taking (3.12) into account, from the first m equation of (3.3). We write these equations using the notation of (3.1), in the form

$$F(\varphi) + Y\psi = \Phi_1[X^\circ(x') + Y^\circ u'] + \Phi_2[X^\circ(x'') + Y^\circ u''] \tag{3.22}$$

$$\Phi_1 = \begin{Vmatrix} \frac{\partial \varphi_1}{\partial x_1'} \cdots \frac{\partial \varphi_1}{\partial x_n'} \\ \cdots\cdots \\ \frac{\partial \varphi_m}{\partial x_1'} \cdots \frac{\partial \varphi_m}{\partial x_n'} \end{Vmatrix}, \quad \Phi_2 = \begin{Vmatrix} \frac{\partial \varphi_1}{\partial x_1''} \cdots \frac{\partial \varphi_1}{\partial x_n''} \\ \cdots\cdots \\ \frac{\partial \varphi_m}{\partial x_1''} \cdots \frac{\partial \varphi_m}{\partial x_n''} \end{Vmatrix}$$

Since according to (3.12) Y has an inverse, we obtain the following explicit formula:

$$\psi = Y^{-1}\{\Phi_1[X^\circ(x') + Y^\circ u'] + \Phi_2[X^\circ(x'') + Y^\circ u''] - F(\varphi)\} \tag{3.23}$$

4. New laws of multiplication by the numbers $\odot_x, \odot_u$. We shall determine the multiplication law $\lambda \odot_x x$ as a function $p(\lambda, x)$, mapping $R \times R^n \to R^n$

$$p(\lambda, x) = \lambda \odot_x x \tag{4.1}$$

We shall seek the multiplication law $\lambda \odot_u u$ in the form of a function $R \times R^m \times R^n \to R^m$

$$q(\lambda, u, x) = \lambda \odot_u u \tag{4.2}$$

Just as in the case of addition, we shall requre the laws (4.1) and (4.2) to transform every solution $x(t)$ of system (1.1) corresponding to the control function $u(t)$, into a new solution $x'(t) = p(\lambda, x(t))$ corresponding to the control function $u'(t) = q(\lambda, u(t), x(t))$. This means that the equation

$$x^{\cdot} = f(x, u) \tag{4.3}$$

must yield

$$p^{\cdot}(\lambda, x) = f(p(\lambda, x), q(\lambda, u, x)) \tag{4.4}$$

On the other hand, we have by virtue of (4.3),

$$p^{\cdot}(\lambda, x) = p_x f(x, u) \tag{4.5}$$

Relations (4.4) and (4.5) yield the fundamental partial differential equation for determining the laws $\odot_x$, $\odot_u$

$$f(p(\lambda, x), q(\lambda, u, x)) = p_x(\lambda, x) f(x, u) \tag{4.6}$$

Whenever it is clear from the text which vectors are multiplied, whether x or u, we shall write instead of $\odot_x$, $\odot_u$, simply $\odot$.

5. Solution of equations determining the laws $\odot_x, \odot_u$. We shall restrict ourselves to considering system (3.1).

Theorem 5.1. Let the function $p(\lambda, x)$ be continuously differentiable in x, and define the multiplication $\lambda \odot x$ satisfying the condition $\lambda^{-1} \odot_x (\lambda \odot_x x) = x$ for $\lambda \neq 0$. Then for $\lambda \neq 0$ we have

$$\operatorname{rank} p_x(\lambda, x) = n \tag{5.1}$$

Proof. Differentiating the equation $p(\lambda^{-1}, p(\lambda, x)) = x$ with respect to x, we obtain $p_x(\lambda^{-1}, p(\lambda, x)) p_x(\lambda, x) = E$, from which (5.1) follows.

We rewrite (4.6) for system (3.1) in the form

$$X^\circ(p) + Y^\circ(p) q = p_x [X^\circ(x) + Y^\circ(x) u] \tag{5.2}$$

Theorem 5.2. The necessary condition for finding a function $p(\lambda, x)$ continuously differentiable in x and a function $q(\lambda, u, x)$ which satisfies (5.2) for all x, λ is, that

$$p_x \Gamma_x \subset \Gamma_p, \quad p_x X^\circ(x) - X^\circ(p) \in \Gamma_p \tag{5.3}$$

for all $x \in R^n$, $\lambda \in R$.

If $p(\lambda, x)$, continuously differentiable in x, defines the multiplication law $\lambda^{-1} \odot_x (\lambda \odot_x x) \equiv x$, $\lambda \neq 0$, $0 \odot_x x = 0$ and $\operatorname{rank} Y^\circ(x) = m$ for all $x \in R^n$, then the first condition of (5.3) is equivalent to

$$p_x \Gamma_x = \Gamma_p, \quad \lambda \neq 0 \tag{5.4}$$

Conversely, if $p(\lambda, x)$ is a function continuously differentiable in x for which conditions (5.3) hold for all $x \in R^n, \lambda \in R$, then a function $q(\lambda, u, x)$ can be found such that p, q satisfy (5.2) for all x, u, λ.

Theorem 5.3. Let $Y(x)$ be constant and let the conditions (3.11) and (3.12) hold. Condition (5.4) for the function $p(\lambda, x)$ differentiable in x and such that $p(0, x) = 0$, is equivalent to the fact that the functions $p_i(\lambda, x)$ with $i \geqslant m+1$ are independent of $x_1, \ldots, x_m$, i.e.

$$p_i(\lambda, x) = p_i(\lambda, x_{m+1}, \ldots, x_n), \quad m+1 \leqslant i \leqslant n \tag{5.5}$$

When (5.5) is satisfied, the second inclusion (5.3) is equivalent to the relation

$$p_x^{(z)}(\lambda, x) Z(x) = Z(p), \quad p_x^{(z)} = \left\| \begin{matrix} \dfrac{\partial p_{m+1}}{\partial x_{m+1}} & \cdots & \dfrac{\partial p_{m+1}}{\partial x_n} \\ \cdot & \cdots & \cdot \\ \dfrac{\partial p_n}{\partial x_{m+1}} & \cdots & \dfrac{\partial p_n}{\partial x_n} \end{matrix} \right\| \tag{5.6}$$

The proofs of Theorems 5.2 and 5.3 are analogous to those of Theorems 3.2 and 3.3.

Let us now specify the law of multiplication $\odot_x$ over the coordinates numbered $m+1$ to n, in the usual manner $p_i(\lambda, x) = \lambda x_i$, $m+1 \leqslant i \leqslant n$. Then Eq.(5.6) will become

$$\lambda Z(x) = Z(p) \tag{5.7}$$

Just as in the case of addition, we introduce the equation

$$\lambda X(x) = X(p(\lambda, x)) \tag{5.8}$$

where $X(x)$ is the same function as in (3.17).

Theorem 5.4. Let the conditions of Theorem 3.4 hold and equation (5.8) have a solution for all $x \in R^n, \lambda \in R$. Then the operations $\oplus_x, \odot_x$ define on R^n the structure of a linear space.

Proof. According to (5.8) we have

$$X(p(\lambda_1, p(\lambda_2, x)) = \lambda_1 X(p(\lambda_2, x)) = \lambda_1\lambda_2 X(x) = X(p(\lambda_1\lambda_2, x))$$

and from this we obtain, according to condition 2 of Theorem 3.4, $p(\lambda_1, p(\lambda_2, x)) = p(\lambda_1\lambda_2, x)$ which can be rewritten in the notation $\odot$ in the form $\lambda_1 \odot (\lambda_2 \odot x) = (\lambda_1\lambda_2) \odot x$. Similarly, we have $1 \odot x = x$.

Let us verify that the axiom of distributivity is satisfied. According to (5.8) and (3.17)

$$X(p(\lambda_1 + \lambda_2, x)) = (\lambda_1 + \lambda_2) X(x) = \lambda_1 X(x) + \lambda_2 X(x) =$$
$$X(p(\lambda_1, x)) + X(p(\lambda_2, x)) = X(\varphi(p(\lambda_1, x), p(\lambda_2, x)))$$

Condition 2 of Theorem 3.4 implies that $p(\lambda_1 + \lambda_2, x) = \varphi(p(\lambda_1, x), p(\lambda_2, x))$, which will be written in the $\oplus, \odot$ notation as $(\lambda_1 + \lambda_2) \odot x = (\lambda_1 \odot x) \oplus (\lambda_2 \odot x)$. According to (5.8) and (3.17)

$$X(p(\lambda, \varphi(x', x''))) = \lambda X(\varphi(x', x'')) = \lambda X(x') + \lambda X(x'') =$$
$$X(p(\lambda, x')) + X(p(\lambda, x'')) = X(\varphi(p(\lambda, x'), p(\lambda, x'')))$$

and this yields, by virtue of condition 2 of Theorem 3.4. $p(\lambda, \varphi(x' + x'')) = \varphi(p(\lambda, x'), p(\lambda, x''))$ or $\lambda \odot (x' + x'') = (\lambda \odot x') \oplus (\lambda \odot x'')$.

We note that condition 4 of Theorem 3.4 is in this case redundant, since it follows from the solvability of (5.8). Moreover, the solvability of (5.8) implies that the equation $X(x_0) = 0$ has a solution. Placing the origin of coordinates at x_0 satisfies condition 3 of Theorem 3.4., therefore only the first two conditions of Theorem 3.4 are essential.

Theorem 5.4 reduces the construction of the law $\odot_x$ to solving the equation (5.8).

Just as for ψ, we can obtain an explicit formula for q

$$q(\lambda, u, x) = Y^{-1}\{P[X^\circ(x) + Y^\circ u] - F(p(\lambda, x))\} \tag{5.9}$$

$$P = \left\| \begin{matrix} \frac{\partial p_1}{\partial x_1} & \cdots & \frac{\partial p_1}{\partial x_n} \\ \cdot & \cdots & \cdot \\ \frac{\partial p_m}{\partial x_1} & \cdots & \frac{\partial p_m}{\partial x_n} \end{matrix} \right\|$$

6. Differentiability of the function φ and p. Since the matrices of partial derivatives in x of the functions φ and p occur in the fundamental equations (2.6) and (4.6), it becomes important to explain the conditions which guarantee the existence of these derivatives.

Theorem 6.1. Let the conditions of Theorem 5.4 hold; the function $X(x)$ is $l \geqslant 1$ times continuously differentiable and for all $x \in R^n$

$$\operatorname{rank} X_x(x) = n \tag{6.1}$$

Then the functions $\varphi(x', x''), p(\lambda, x)$, specified as solutions of equations (3.17), (5.8), are l times continuously differentiable in all variables, and the following relations hold:

$$\varphi(x', x'') = x' + x'' + 0(\|x'\| + \|x''\|) \tag{6.2}$$
$$p(\lambda, x) = \lambda x + \lambda\alpha(x) + 0(\lambda[x + \alpha(x)]) \tag{6.3}$$

Here $\alpha(x) = 0(\|x\|)$, $\lim \varepsilon^{-1} 0(\varepsilon) = 0 \ (\varepsilon \to 0)$.

Proof. The continuous differentiability of φ and p follows from the theorem on an implicit function. We shall prove (6.3). From (5.8) and the differentiability of the solution of $X(p) = r$, $r = \lambda X(x)$ it follows that

$$p(\lambda, x) = X_x^{-1}(0) r + 0(r) = X_x^{-1}(0)\lambda X(x) + 0(\lambda X(x)) =$$
$$\lambda X_x^{-1}(0)[X_x(0) x + \theta(x)] + 0([\lambda X_x(0) x + \lambda\theta(x)]) =$$
$$\lambda x + \lambda X_x^{-1}(0)\theta(x) + 0_1(\lambda[x + X_x^{-1}(0)\theta(x)]) \tag{6.4}$$

and from this (6.3) follows. We prove (6.2) in the same manner.

Henceforth, we shall require two mappings $R^n \to R^n$ defined by the formulas

$$y = \varkappa(x) = \lim_{\lambda\to 0} \lambda^{-1} p(\lambda, x); \quad x = s(y) = \lim_{\lambda\to 0} p(\lambda^{-1}, \lambda y) \tag{6.5}$$

Theorem 6.2. Let the conditions of Theorem 6.1 and condition

$$\| X(x) \| \to \infty \quad \text{as} \quad \| x \| \to \infty \tag{6.6}$$

all hold.

Then the limits (6.5) exist; the functions $\varkappa(x)$ and $s(y)$ are l times continuously differentiable, the mappings $\varkappa, s$ define the isomorphisms of the space R^n with operations $\oplus_x, \odot_x$ and R^n with the usual operations of addition $+$ and multiplication by a number; the mappings $\varkappa$ and s are defined uniquely by the equations

$$X_\varkappa(0)\, \varkappa(x) = X(x) \tag{6.7}$$

$$X(s(y)) = X_x(0)\, y \tag{6.8}$$

and the following relations hold:

$$s(\varkappa(x)) \equiv x, \quad \varkappa(s(y)) \equiv y \tag{6.9}$$

$$\varkappa_i(x) = x_i, \quad s(y) = y_i, \quad m + 1 \leqslant i \leqslant n \tag{6.10}$$

Proof. From (6.1) and (6.4) we obtain (6.7), and this implies the differentiability of $\varkappa$. By virtue of Theorem 5.4 $\oplus_x, \odot_x$ define a linear space on R^n. By definition, p and φ are solutions of (3.17) and (5.8), therefore according to (6.7),

$$X_x(0)\, \varkappa(x' \oplus x'') = X(x' \oplus x'') = X(x') + X(x'') = X_x(0)\, \varkappa(x') + X_x(0)\, \varkappa(x''); \quad X_x(0)\, \varkappa(\lambda \odot x) = X(\lambda \odot x) = \lambda X(x) = X_x(0)\, \lambda \varkappa(x)$$

This implies, by virtue of condition (6.1), that

$$\varkappa(x' \oplus x'') = \varkappa(x') + \varkappa(x''); \quad \varkappa(\lambda \odot x) = \lambda \varkappa(x)$$

i.e. $\varkappa$ is a homomorphism of vector spaces.

According to (5.8) we have $X(p(\lambda^{-1}, \lambda y)) = \lambda^{-1} X(\lambda y)$, $\lambda \neq 0$, which yields

$$X(p(\lambda^{-1}, \lambda y)) = X_x(0)\, y + \lambda^{-1} 0(\lambda y) \tag{6.11}$$

Relation (6.6) and the boundedness of $\lambda^{-1} 0(\lambda y)$ as $\lambda \to 0$ together imply the boundedness of $p(\lambda^{-1}, \lambda y)$ as $\lambda \to 0$. In this case, if the limit $\lim p(\lambda^{-1}, \lambda y)$ did not exist as $\lambda \to 0$, then we could find two different limit points as $\lambda \to 0$, $x' \neq x''$ of the curve $p(\lambda^{-1}, \lambda y)$. From (6.11) we obtain $X(x') = X(x'') = X_x(0)\, y$ and this yields, according to condition 2 of Theorem 3.4, $x' = x''$. The contradiction shows that a second limit (6.5) also exists. From (6.11) we obtain (6.8).

From (3.17) and (6.8) it follows that

$$X(s(y') \oplus s(y'')) = X(s(y')) + X(s(y'')) = X_x(0)(y' + y'')$$

while from (5.8) and (6.8) we obtain

$$X(\lambda \odot s(y)) = \lambda X(s(y)) = \lambda X_x(0)\, y = X_x(0)(\lambda y)$$

When we take condition 2 of Theorem 3.4 into account, the above equations mean that the elements $y' + y''$, λy have the corresponding elements $s(y') \oplus s(y'')$ and $\lambda \odot s(y)$, i.e. s is a homomorphism of vector spaces. The unique solvability of (6.7) in $\varkappa(x)$ and of (6.8) in $s(y)$ implies that $\varkappa(s(y)) \equiv y$, $s(\varkappa(x)) \equiv x$, i.e $\varkappa$ and s are isomorphisms.

7. Synthesis of automatic control systems based on the isomorphism of dynamic control systems. Below we shall understand by W the set of pairs of functions $x(t)$, $u(t)$ which satisfy (1.1) and belong to some, previously discussed class. It is understood that these classes may be represented by:

1) continuous control functions $u(t)$ and continuously differentiable trajectories $x(t)$;

2) piecewise continuous $u(t)$ and piecewise continuously differentiable $x(t)$;

3) Lebesque-summable controls $u(t)$ and absolutely continuous trajectories $x(t)$ satisfying (1.1) almost everywhere.

The operations $\oplus_x, \oplus_u, \odot_x, \odot_u$ on W serve to introduce the structure of linear space according to the formulas (2.7), (2.8) for addition, and similarly for multiplication. The structure is in agreement with the dynamics of the system, i.e is one in which the system is linear.

Suppose we have two systems of the form (1.1) and linear spaces W^1 and W^2 for them. We shall say that the systems are isomorphic, if the linear spaces W^1 and W^2 are isomorphic. As a special case of this we take a single system (1.1) and $W^1 = W^2 = W$, and the automorphism of the linear space W serves as the isomorphism. Here the most important automorphism is that of multiplication $\lambda \odot$ by a number $\lambda \neq 0$.

Using the concept of isomorphism we can find, for the initial system, a simple isomorphic system for use in constructing a regular or a programmed control with the required properties.

Then, if the isomorphism itself is free of any pathologies, hopefully the regular or programmed control, converted through the isomorphism to the initial system, has the same properties. Specific conclusions can be drawn by inspecting the mapping defining the given isomorphism.

The pairs of functions from W^1 and W^2 which can be transformed into each other by the isomorphism, will be called isomorphic with respect to this isomorphism. We shall call a certain property of the pair $x^{(1)}(t)$, $u^{(1)}(t)$ from W^1 invariant with respect to the isomorphism in question, if its presence in $x^{(1)}, u^{(1)}$ implies its presence in the isomorphic pair $x^{(2)}(t)$, $u^{(2)}(t)$ from W^2.

Consider system (3.1), (3.11) and a system of linear approximation for it

$$w^{\cdot} = F_x(0)\, y + Yv \tag{7.1}$$
$$z^{\cdot} = Z_x(0)\, y, \quad y = \begin{Vmatrix} w \\ z \end{Vmatrix}, \quad w \in R^m, \ z \in R^{n-m}, \ v \in R^m$$

To construct the isomorphism of systems (3.1) and (7.1), we shall require the equation

$$X_x(x)[X^{\circ}(x) + Y^{\circ}u] = X_x(0) \begin{Vmatrix} F_x(0)\, y + Yv \\ Z(x) \end{Vmatrix} \tag{7.2}$$

We shall denote the solution of the equation (7.2) in v and u by $v = v(x, y, u)$ and $u = u(x, y, v)$ respectively, and define the isomorphisms of systems (3.1) and (7.1) as the mappings

$$x(t), \quad u(t) \to y(t), v(t);\ y(t) = \varkappa(x(t)),\ v(t) = v(x(t), \quad \varkappa(x(t)),\ u(t)) \tag{7.3}$$
$$y(t), v(t) \to x(t), \quad u(t);\ x(t) = s(y(t)), \quad u(t) = u(s(y(t)), \quad y(t), v(t)) \tag{7.4}$$

i.e. (7.3) or (7.4) is a mapping of the pair of vectors $x, u\ (y, v)$ into the pair of vectors $y, v\ (x, u)$ which induces the mapping of the vector space of function pairs $W\ (W^{\circ})$ corresponding to system (3.1) (or to system (7.1)) into $W^{\circ}\ (W)$.

We shall denote by $D(y)$ the locally Lipshitz function $R^n \to R^m$.

Theorem 7.1. Let the conditions of Theorem 6.2 hold, and let the function $X^{\circ}(x)$ be continuously differentiable. When $n < 2(n-m)$, we also assume that the solution (3.17) satisfies (3.16) and the solution (5.8) satisfies (5.7). Then

1) the mapping $x(t) \to \lambda \odot_x x$; $u(t) \to \lambda \odot_u u(t)$ with $\lambda \neq 0$ is an automorphism of the system (3.1), (3.11);

2) the solutions $v(x, y, u)$, $u(x, y, v)$ of system (7.2) exist, are defined uniquely, and are continuous functions, and mappings (7.3), (7.4) are isomorphisms of systems (3.1), (3.11) and (7.1);

3) asymptotic stability in toto of system (7.1) with the regulator $v = D(y)$ is equivalent to the asymptotic stability in toto of the system (3.1) with the regulator $u = u(x, \varkappa(x), D(\varkappa(x)))$ obtained with help of isomorphism (7.4);

4) the following properties are invariant with respect to the automorphism $\lambda \odot$ and isomorphisms (7.3) and (7.4): a) the trajectory tends to the origin of coordinates as $t \to \infty$, and b) the trajectory reaches the origin of coordinates after the time T.

Proof. From (6.7) it follows that

$$X_x(0)\, \varkappa_x(x) = X_x(x)$$

By virtue of (6.1) we find, that (7.2) is equivalent to

$$\varkappa_x(x)[X^{\circ}(x) + Y^{\circ}u] = \begin{Vmatrix} F_x(0)\, y + Yv \\ Z(x) \end{Vmatrix} \tag{7.5}$$

From (6.10), (3.11), (3.12) and the invertibility of $\varkappa_x(x)$ it follows that (7.5), and therefore (7.2), have unique solutions in v or u.

From (5.7) and the relation $0 \odot x = 0$ it follows that $Z(0) = 0$. The differentiability of $X^{\circ}(x)$ yields $Z(p) = Z_x(0)\, p + 0(p)$. Then from (5.7), (6.3), (6.5), (6.9) we obtain

$$Z_x(0)\, y = Z(x) \tag{7.6}$$

We shall show that the mapping (7.3) transforms the pairs $x(t), u(t)$ satisfying (3.1) into the pairs $y(t), v(t)$ satisfying (7.1). Since $y = \varkappa(x)$, we obtain from (7.3)

$$y^{\cdot}(t) = \varkappa_x(x)\, x^{\cdot}(t) = X_x^{-1}(0)\, X_x(0)\, \varkappa_x(x)\, x^{\cdot}(t) = X_x^{-1}(0)\, X_x(x)\, x^{\cdot}(t)$$

This yields, by virtue of (3.1), (7.2), (7.6),

$$y^{\cdot}(t) = X_x^{-1}(0)\, X_x(x)[X^{\circ}(x) + Y^{\circ}u] = \begin{Vmatrix} F_x(0)\, y + Yv \\ Z(x) \end{Vmatrix} = \begin{Vmatrix} F_x(0)\, y + Yv \\ Z_x(0)\, y \end{Vmatrix}$$

which is identical with (7.1).

Next we shall show that (7.3) defines the homomorphism of the vector spaces $W \to W^\circ$. Let us check the property of homomorphism with respect to multiplication by a number. Let the mapping (7.3) transform the vectors x, u into y, v. Then $\lambda \odot_x x, \lambda \odot_u u$ transform (7.3) into some vectors y', v'. By virtue of Theorem 6.2 we have $y' = \lambda y$. It remains to show that $v' = \lambda v$. From (7.2) it follows that

$$X_x(\lambda \odot x)[X^\circ(\lambda \odot x) + Y^\circ(\lambda \odot u)] = X_x(0)\begin{Vmatrix} F_x(0)y' + Yv' \\ Z(\lambda \odot x) \end{Vmatrix}$$

Then, taking (5.7) and $y' = \lambda y$ into account we obtain

$$X_x(\lambda \odot x)[X^\circ(\lambda \odot x) + Y^\circ(\lambda \odot u)] = X_x(0)\begin{Vmatrix} \lambda F_x(0)y + Yv' \\ \lambda Z(x) \end{Vmatrix} \tag{7.7}$$

On the other hand, (5.8) implies that $X_x(\lambda \odot x)\, p_x(\lambda, x) = \lambda X_x(x)$. Multiplying (7.2) by λ and taking (4.6) into account, we obtain

$$X_x(0)\begin{Vmatrix} \lambda F_x(0)y + \lambda Yv \\ \lambda Z(x) \end{Vmatrix} = \lambda X_x(x)[X^\circ(x) + Y^\circ u] =$$
$$X_x(\lambda \odot x)\, p_x(\lambda, x)[X^\circ(x) + Y^\circ u] = X_x(\lambda \odot x)[X^\circ(\lambda \odot x) + Y^\circ(\lambda \odot u)]$$

Combining this expression with (7.7) we obtain, by virtue of (6.1) and (3.12), $v' = \lambda v$, which we have set out to show. The operation of addition is confirmed in exactly the same manner. Thus (7.3) is a homomorphism of $W \to W^\circ$. We can show in the same manner that (7.4) is also a homomorphism of $W^\circ \to W$. The unique solvability of (7.2) and the identities $s(\varkappa(x)) \equiv x$, $\varkappa(s(y)) \equiv y$ together imply that (7.3) and (7.4) are mutually inverse isomorphisms. The remaining assertions of the theorem follow from the differential properties of the isomorphism $\varkappa$.

Let us see how we can use the automorphism $\lambda \odot$ to solve the problem of synthesizing the regulator. We shall write, for convenience, $x_\lambda = \lambda \odot_x x$, $u_\lambda = \lambda \odot_u u$. We shall assume that the regulator $u = D(x)$ of system (3.1) is already constructed in the neighbourhood of the state of equilibrium $x = 0$, and ensures the asymptotic stability of the state $x = 0$. Let $x(t)$ be the running state of the system. We extend the regulator $u = D(x)$ to embrace the state $x(t)$ also, and choose $\lambda \neq 0$ to have a sufficiently small numerical value so that the vector $x_\lambda(t)$ lies within the zone of action of the regulator $u = D(x)$. The resulting isomorphic trajectories $x_\lambda(t)$ have the corresponding isomorphic controls $u_\lambda = D(x_\lambda(t))$ and

$$u(t) = \lambda^{-1} \odot_u D(x_\lambda(t)) = \lambda^{-1} \odot_u D(\lambda \odot_x x(t)) \tag{7.8}$$

Since the regulator $D(x)$ ensures the asymptotic stability of the equilibrium state, it follows that $x_\lambda(t) \to 0$ as $t \to \infty$. Then we obtain for the continuous function $p(\lambda, x)$, $p(\lambda, 0) = 0$, $x(t) = \lambda^{-1} \odot_x x_\lambda(t) = p(\lambda^{-1}, x_\lambda(t)) \to 0$ as $t \to \infty$. The quantity λ is chosen to be small. Therefore the following passage to the limit in (7.8) appears natural:

$$u(x) = \lim_{\lambda \to 0} \lambda^{-1} \odot_u D(\lambda \odot_x x) \tag{7.9}$$

Let us find the relation between the regulator (7.9) and the regulator obtained from the isomorphism of system (3.1) with its linear part (7.1) determined by equation (7.2). According to (5.2) we have

$$X^\circ(p(\lambda^{-1}, x_\lambda)) + Y^\circ q(\lambda^{-1}, x_\lambda, u_\lambda) = p_x(\lambda^{-1}, x_\lambda)[X^\circ(x_\lambda) + Y^\circ u_\lambda] \tag{7.10}$$

Since $p(\lambda^{-1}, x_\lambda) = \lambda^{-1} \odot \lambda \odot x = x$, then multiplying (7.10) on the left by $X_x(x)$, and using the relation $\lambda^{-1}X_x(x_\lambda) = X_x(x)\, p_x(\lambda^{-1}, x_\lambda)$, which follows from (5.8) we obtain

$$X_x(x)[X^\circ(x) + Y^\circ u] = \lambda^{-1}X_x(x_\lambda)[X^\circ(x_\lambda) + Y^\circ u_\lambda]$$

Substituting here the regulator $u_\lambda = D(x_\lambda(t))$ we obtain, provided that the limit $\lim \lambda^{-1}D(\lambda \odot x)$ $(\lambda \to 0)$ exists, taking (6.5) and the fact that $x_\lambda \to 0$ as $\lambda \to 0$ into account,

$$X_x(x)[X^\circ(x) + Y^\circ u] = X_x(0)[X_x^\circ(0)\, y + Y^\circ \lim_{\lambda \to 0} \lambda^{-1}D(\lambda \odot x)]$$

where

$$\lim \lambda^{-1}X^\circ(x_\lambda) = \lim \lambda^{-1}X_x^\circ(0)\, x_\lambda = \lim X_x^\circ(0)\, \lambda^{-1}p(\lambda, x) = X_x^\circ(0)\, y \quad (\lambda \to 0).$$

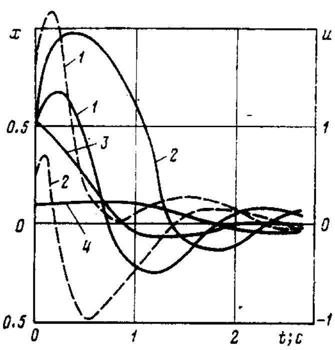
Comparing this equation with (7.2) we find, that when $v = \lim \lambda^{-1}D(\lambda \odot x)$ $(\lambda \to 0)$, the equation is identical with (7.2). Therefore the regulator (7.9) can be expressed in terms of the solution $u(x, \varkappa(x), v)$ of (7.2) in the form

$$u(x) = u\left(x, \varkappa(x), \lim_{\lambda\to 0} \lambda^{-1} D(\lambda \odot x)\right) \tag{7.11}$$

8. Example. Let us consider the controlled rotation of a satellite whose dynamics is described by the system /1/

$$x^{\cdot} = X^{\circ}(x) + u_1 Y_1^{\circ} + u_2 Y_2^{\circ}, \quad \omega(x) = [1-(x_3^2+x_4^2)]^{1/2} \tag{8.1}$$

$$x = \begin{Vmatrix} x_1 \\ \cdots \\ x_4 \end{Vmatrix}, \quad X^{\circ}(x) = \begin{Vmatrix} -2x_2 \\ 2x_1 \\ -x_2\omega(x) + x_4 \\ x_1\omega(x) - x_3 \end{Vmatrix}, \quad Y_1^{\circ} = \begin{Vmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Vmatrix}, \quad Y_2^{\circ} = \begin{Vmatrix} 0 \\ 1 \\ 0 \\ 0 \end{Vmatrix}$$



The coordinates x_1 and x_2 are components of the spin vector, and x_3, x_4 are components of the auxiliary vector of unit length, which arise when we pass from the coordinate system attached rigidly to the satellite, to the inertial system. Here u_1 and u_2 denote the momenta generated by two rocket motors. We pose the problem of stabilizing the rotation, i.e. the rotation of the satellite about a fixed axis corresponding to $x = 0$ is the desired state. We obtain the standard problem of stabilizing the zero state. Let us pose the problem of constructing a regulator $u = u(x)$ ensuring the asymptotic stability in the large.

We note that x_3, x_4 satisfies the inequality $x_3^2 + x_4^2 \leqslant 1$ by definition, therefore we shall limit ourselves to this region.

Let us find $X(x)$ using (3.18), (3.19) as $X_1 = -x_2\omega(x) + x_4$, $X_2 = x_1\omega(x) - x_3$, $X_3 = x_3$, $X_4 = x_4$. The conditions of Theorem 7.1 are satisfied and equations (3.16) take the form

$$-x_2'\omega(x') + x_4' - x_2''\omega(x'') + x_4'' = -(x' \oplus x'')_2\,\omega(x' \oplus x'') + (x' \oplus x'')_4$$
$$x_1'\omega(x') - x_3' + x_1''\omega(x'') - x_3'' = (x' \oplus x'')_1\,\omega(x' \oplus x'') - (x' \oplus x'')_3$$

Remembering that according to (3.15) $(x' \oplus x'')_3 = x_3' + x_3''$, $(x' \oplus x'')_4 = x_4' + x_4''$, we obtain the addition laws for the first two components of the state vectors

$$(x' \oplus x'')_k = \omega^{-1}(x' \oplus x'')\,[x_k'\omega(x') + x_k''\omega(x'')]; \quad k = 1, 2$$
$$\omega(x' \oplus x'') = [1 - (x_3' + x_3'')^2 - (x_4' + x_4'')^2]^{1/2}$$

Similarly, from (5.8) we obtain the multiplication law

$$\lambda \odot x_k = [1 - \lambda^2(x_3^2 + x_4^2)]^{-1/2}\lambda x_k \omega(x); \quad k = 1, 2; \quad \lambda \odot x_i = \lambda x_i;\ i = 3, 4$$

The mapping (6.5) $x \to \varkappa(x)$ will have the form

$$y_k = \varkappa_k(x) = \lim_{\lambda\to 0} \lambda^{-1}(\lambda \odot x)_k = x_k\omega(x); \quad k = 1, 2; \quad y_3 = x_3; \quad y_4 = x_4$$

The linear approximation system (7.1) will be written for (8.1) in the form

$$y_1^{\cdot} = -2y_2 + v_1, \quad y_2^{\cdot} = 2y_1 + v_2$$
$$y_3^{\cdot} = -y_2 + y_4, \quad y_4^{\cdot} = y_1 - y_3 \tag{8.2}$$

We choose a regulator for this system, ensuring the asymptotic stability, in the form

$$v_1 = D_1(y) = -2y_1 + ay_3 + by_4; \quad v_2 = D_2(y) = -2y_2 + cy_3$$
$$a = 3\sqrt{5}; \quad b = -3{,}75; \quad c = 3{,}75 \tag{8.3}$$

Solving (7.2) we obtain the regulator isomorphic to the regulator (8.3)

$$u_1 = -2x_1 + \omega^{-1}(x)\,(ax_3 + bx_4 - x_1x_2x_3 + x_1^2x_4)$$
$$u_2 = -2x_2 + \omega^{-1}(x)\,(cx_3 + x_1x_2x_4 - x_2^2x_3) \tag{8.4}$$

and the same regulator can also be obtained according to (7.9), when

$$D_1(\lambda \odot_x x) = -2(\lambda \odot x)_1 + a(\lambda \odot x)_3 + b(\lambda \odot x)_4; \; D_2 = -2(\lambda \odot x)_2 + c(\lambda \odot x)_3$$

The results of modelling system (8.1) with the regulator (8.4) obtained are shown in the figure. The solid curves *1–4* correspond to x_1, x_2, x_3, x_4. The dashed curves *1, 2* depict u_1 and u_2. The results of modelling under varying initial conditions lead to the conclusion that the closed system obtained is stable in the large and the nature of the transient for it is identical with those for the linearized system (8.2).

We note that sometimes it is best to reduce the dimensionality of the problem in question /2/.

If $Y(x)$ depends continuously on x and (3.6) holds everywhere, then replacing the control vector u by $Y(x)$ we can reduce the system to the form (3.1), (3.11), i.e. the dependence of

$Y(x)$ on x is no hindrance in the method proposed.

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ON THE STUDY OF MINIMAX EVALUATION OF PARAMETERS OF NON-LINEAR SYSTEMS *

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The problem of the a posteriori minimax evaluation /1/ of the unknown parameters of non-linear systems of fairly general form is investigated. Approximations of information sets related to the observation process are defined using the non-linear theory of duality. The asymptotic properties of minimax estimates are also obtained in the case of perturbations that can be represented in the form of random processes. Problems of minimax observation as applied to non-linear systems were investigated in /2, 3/.

1. Let us assume that the observed signal conforms to the equation

$$y(t) = g(t, z, w(t)), \quad t \in [0, T] \tag{1.1}$$

where the unknown vector of the parameters $z \in R^n$ and perturbation $w(T; \cdot) = \{w(t), t \in [0, T]\}$ satisfy constraints of the form

$$z \in Z^\circ, \quad w(t) \in W(t) \subseteq W, \quad t \in [0, T]$$

The m-vector of the function $g(\cdot, \cdot, \cdot)$ and the input data are assumed to be as follows

1) Z° and W are compact in R^n and R^s, respectively;

2) $g(t, z, w)$ is continuous with respect to the set of variables and, moreover, the set of functions $\{g(t, z, \cdot), t \geqslant 0\}$ is equicontinuous for any $z \in Z^\circ$;

3) the class of admissible perturbations is defined by the set

$$E = \{w(T; \cdot) \in C^s[0, T]: w(t) \in W(t), \ t \in [0, T]\}$$

4) the set of possible outputs of system (1.1)

$$G(z) = \{f(T; \cdot): f(t) = g(t, z, w(t)); \quad w(T; \cdot) \in E\} \tag{1.2}$$

is closed in space $C^m[0, T]$.

Definition 1.1. The set

$$Z(T; y(\cdot)) = \{z': y(T; \cdot) \in G(z')\}$$

is called the information set, compatible with the signal $y(T; \cdot) = \{y(t), t \in [0, T]\}$.

Points $z_*(T) \in Z(T; y(\cdot))$, somehow separated, will be called the a posteriori minimax evaluations of the vector of parameters z.

To describe the weak dependence of random processes, which simulate perturbations in a stochastic system, we use following definition.

Definition 1.2. The random process $\{w(t), t \geqslant 0\}$ in the probability space $\{\Omega, \Sigma, P\}$ with the phase space $\{R^s, \Delta\}$ is called entirely regular, if

$$\alpha(\tau) = \sup_{A \in \Gamma_0^t,\, B \in \Gamma_{t+\tau}^\infty,\, t \geqslant 0} |P(AB) - P(A)P(B)| \to 0$$

as $\tau \to \infty$, where Γ^b_a, $0 \leqslant a \leqslant b \leqslant +\infty$, is the σ-algebra generated by $\{w(t), a \leqslant t \leqslant b\}$.

We will present without proof the statement that defines the entirely regular processes.

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